→ j^{th} component of $|\alpha\rangle$

$$\rightarrow \langle j|\alpha\rangle = \sum_i a_i \underbrace{\langle j|i\rangle}_{\delta_{ji}} = \underbrace{\sum_i a_i \delta_{ji}}_{a_j}$$

*D6. subspace : For a vector space V. a subset that forms a vector space, denoted as $V_i^{n_i}$ (Dimension; subspace)

e.g. $V^3_{xyz}(\mathbb{R})$ $\begin{cases} V_x^1 \\ V_y^1 \\ V_z^1 \end{cases}$

*D7 sum of subspace $\oplus$: $V_i^{n_i} \oplus V_j^{n_j} = V_k^{n_k}$

$V_k^{n_k}$ contains $\begin{cases} ① \ \{i\} \ \forall i \text{ in } V_i^{n_i} \\ ② \ \{j\} \ \forall j \text{ in } V_j^{n_j} \\ ③ \text{ linear combinations of } \{i\} \ \& \ \{j\} \end{cases}$

e.g. $V_x^1 \oplus V_y^1 = V_{xy}^2$

3> Operator (in function space e.g. $\hat{A} = \frac{\partial}{\partial x}$)

↓

$\hat{X} \doteq$ observables that can act on ket produce another ket vector

$$\hat{X}|\alpha\rangle = |\alpha'\rangle$$

$$\hookrightarrow \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} = a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + b\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

① addition.

* commutative $\hat{X}+\hat{Y}=\hat{Y}+\hat{X}$

* associative $(\hat{X}+\hat{Y})+\hat{Z}=\hat{X}+(\hat{Y}+\hat{Z})$

② multiplication

* non-commutative : $\hat{X}\hat{Y}\neq\hat{Y}\hat{X}$ (typically)

* associative : $(\hat{X}\hat{Y})\hat{Z}=\hat{X}(\hat{Y}\hat{Z})$

* Power : $(\hat{X})^n(\hat{X})^m=(\hat{X})^{n+m}$

③ Linearity :

$$\hat{X}(c_\alpha|\alpha\rangle+c_\beta|\beta\rangle)=c_\alpha\hat{X}|\alpha\rangle+c_\beta\hat{X}|\beta\rangle$$

④ $\langle\beta|\underbrace{\hat{X}|\alpha\rangle}_{|\alpha'\rangle}\in\mathbb{C}$

⑤ $\underbrace{\hat{X}}_{2.}\underbrace{\hat{Y}}_{1.}|\alpha\rangle=\hat{X}|\hat{Y}\alpha\rangle$

⑥ commutation : $[\hat{X},\hat{Y}]=\hat{X}\hat{Y}-\hat{Y}\hat{X}$

$$[\hat{X}\hat{Y},\hat{Z}]=\hat{X}[\hat{Y},\hat{Z}]+[\hat{X},\hat{Z}]\hat{Y}$$

* { * order
* chain rule

$$[\hat{X},\hat{Y}\hat{Z}]=\hat{Y}[\hat{X},\hat{Z}]+[\hat{X},\hat{Y}]\hat{Z}$$

Projection operator.

$$|\alpha\rangle = \sum_i \underbrace{a_i}_{\langle i|\alpha\rangle \text{ scalar}} |i\rangle = \sum_i \underbrace{|i\rangle\langle i|}_{\mathbb{1} = \sum_i \Lambda_i} |\alpha\rangle$$

$$\mathbb{1} = \underbrace{\sum_i \Lambda_i}_{\text{closure / Completeness}}$$

$$\underbrace{\Lambda_i}_{\text{Projection operator.}} |\alpha\rangle = \underbrace{|i\rangle\langle i|}_{\text{Projection.}} \alpha\rangle = a_i |i\rangle$$

$$\langle\alpha|\Lambda_i = \langle \Lambda_i \alpha| = \langle a_i \cdot i| = a_i^* \langle i|$$

$$\Lambda_i \Lambda_j = |i\rangle\underline{\langle i|j\rangle}\langle j| = \begin{cases} i \neq j & 0 \\ i = j & \delta_{ij} \Lambda_j \end{cases}$$

Ajoint

$$|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|$$

$$a|\alpha\rangle = |a\alpha\rangle \xleftrightarrow{DC} \langle a\alpha| = a^* \langle\alpha|$$

$$\hat{x}|\alpha\rangle \xleftrightarrow{DC}$$

$$\|$$

$$|\hat{x}\alpha\rangle \xleftrightarrow{DC} \langle \hat{x}\alpha| \neq \langle\alpha|\hat{x}$$

$$\Downarrow$$

$$\langle\alpha|\hat{x}^+$$

def. $\langle\alpha|\hat{x}^+ \equiv \langle\hat{x}\alpha|$

$$\langle \underbrace{\hat{x}\beta}_{\beta'} |\alpha\rangle = \underline{\langle\beta|\hat{x}^+|\alpha\rangle} = \langle\beta'|\alpha\rangle = (\langle\alpha|\beta'\rangle)^*$$

$$= (\langle\alpha|\hat{x}\beta\rangle)^*$$

$\Rightarrow \langle\beta|\hat{X}^{+}|\alpha\rangle = (\langle\alpha|\hat{X}|\beta\rangle)^{*}$

$= (\langle\alpha|\hat{X}|\beta\rangle)^{*}$

$$\underbrace{\langle i|\hat{X}^{+}|j\rangle}_{(\hat{X}^{+})_{ij}} = \underbrace{(\langle j|\hat{X}|i\rangle)^{*}}_{(\hat{X}_{ji})^{*}}$$

$\boxed{\text{$\hat{X}^{+}$: transpose. complex conjugate of $\hat{X}$}}$

$$\Rightarrow (\hat{X}\hat{Y})^{+} = \hat{Y}^{+}\hat{X}^{+}$$

(Proven by $\langle\hat{X}\hat{Y}\alpha|$)

---

matrix element

matrix element i^{th} row j^{th} col.

$$\hat{X} = \sum_i \sum_j \underbrace{|i\rangle\langle i|}_{\mathbb{1}} \hat{X} \underbrace{|j\rangle\langle j|}_{\mathbb{1}}$$

Hermitian (& Anti-Hermitian) operator.

*D8. Hermitian: $\hat{X}^{+} = \hat{X}$

$\Rightarrow$ ① $X_{ii} = X_{ii}^{*}$

$\Rightarrow$ diag must be real.

$\Rightarrow$ ② $X_{ij} = X_{ji}^{*}$

complex conjugate symmetry

*D9. Anti-Hermitian:

$\hat{X}^{+} = -\hat{X}$

e.g. Any $\hat{X} \to$ $$\hat{X} = \underbrace{\frac{\hat{X}+\hat{X}^{+}}{2}}_{\hat{Y}\ \text{Herm-}} + \underbrace{\frac{\hat{X}-\hat{X}^{+}}{2}}_{\hat{Z}\ \text{Anti-Hermi-}}$$

* T2: 2-1

eigenvalue of $\hat{x}_H$ is real

2-2.

eigenket of $\hat{x}_H$ corresponding to different eigenvalues are orthogonal.

Unitary operator.

* D10 unitary : $\underline{\hat{U}^+\hat{U} = \hat{U}\hat{U}^+} = \mathbb{1}$ → 2x2 rotation matrix in 2D complex space.

$$\hat{U} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \hat{U}^+ = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$$

SU(2)

$$\Rightarrow \hat{U} = \begin{pmatrix} e^{i\phi_1}\cos\theta & -e^{i(\phi_1+\pi)}\sin\theta \\ e^{i\phi_2}\sin\theta & e^{i(\phi_2+\pi)}\cos\theta \end{pmatrix}$$

$$\hat{U} = \begin{bmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{bmatrix}$$

rotation around z-axis w/ φ

if $\phi_1 = \phi_2 = 0$

$$\Rightarrow \hat{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

*. T3. For orthonormal complete basis set.

$\{|a_i\rangle\}$ $\{|b_i\rangle\}$ that span the same vector space

that exists an $\hat{U}$. s.t. $|b_i\rangle = \hat{U}|a_i\rangle \ \forall i$

$i \in \{1, 2, \cdots n\}$

Transformation matrix can transform the vector representation in one basis set into another.