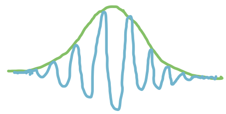
4-4. Gaussian wave packet

$$\langle x'|\alpha\rangle = \psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}}\, \underbrace{e^{\left(ikx' - \frac{x'^2}{2d^2}\right)}}_{e^{ikx'}\, e^{-\frac{x'^2}{2d^2}}}$$



1> Plane wave (due to $e^{ikx'}$)

2> modulated by Gaussian Profile ($e^{-\frac{x'^2}{2d^2}}$)

3> centered at origin.

4> Probability density $|\langle x'|\alpha\rangle|^2$ has a Gaussian shape w/ width of d.

---

* compute $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{p}^2\rangle$

1> $\hat{x}|x'\rangle = x'|x'\rangle$

$$\langle\hat{x}\rangle_\alpha = \langle\alpha|\hat{x}|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx' \,\langle\alpha|x''\rangle \underbrace{\langle x''|\overbrace{\hat{x}|x'\rangle}^{x'|x'\rangle}}_{x'\langle x''|x'\rangle}\langle x'|\alpha\rangle$$

$\mathbb{1}$ $\qquad$ $\mathbb{1}$

$x'\langle x''|x'\rangle$

$$\delta(x''-x') \begin{cases} =0 & \text{if } x''\neq x' \\ =\tilde{1} & \text{if } x''=x' \end{cases}$$

$$= \int_{-\infty}^{\infty} dx' \,\langle\alpha|x'\rangle\, x' \langle x'|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} dx' \,\underbrace{|\langle x'|\alpha\rangle|^2}_{\text{even function.}}\, \underbrace{x'}_{\text{odd function.}} = 0$$

2> $\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} dx' \, |\langle x' | \alpha \rangle|^2 \, x'^2$

$$= \int_{-\infty}^{\infty} dx' \, \overset{\psi_\alpha^*(x')}{\langle \alpha | x' \rangle} \overset{\psi_\alpha(x')}{\langle x' | \alpha \rangle} \, x'^2$$

$\leftarrow$ Gaussian wave packet.

$$= \int_{-\infty}^{\infty} dx' \, \frac{1}{\sqrt{\pi}\, d} \left[e^{\left(-\cancel{ik}x' - \frac{x'^2}{2d^2}\right)} \right] \left[e^{\left(i\cancel{k}x' - \frac{x'^2}{2d^2}\right)} \right] \cdot x'^2$$

$$= \int_{-\infty}^{\infty} dx' \, \frac{1}{\sqrt{\pi}\, d} \, e^{-\frac{x'^2}{d^2}} \cdot x'^2 = \frac{d^2}{2}$$

$$\Rightarrow \langle (\Delta \hat{x})^2 \rangle = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = \frac{d^2}{2} - 0 = \frac{d^2}{2}$$

(std. dev. $\delta \hat{x} = \sqrt{\langle \Delta \hat{x}^2 \rangle}$)

3> $\hat{p} = -i\hbar \frac{\partial}{\partial x'}$

$$\Rightarrow \langle \hat{p} \rangle_\alpha = \langle \alpha | \hat{p} | \alpha \rangle = \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx' \, \langle \alpha | x'' \rangle \underbrace{\langle x'' | \hat{p} | x' \rangle}_{-i\hbar \frac{\partial}{\partial x'} \delta(x'' - x')} \langle x' | \alpha \rangle$$

$$= \int_{-\infty}^{\infty} dx' \, \underbrace{\langle \alpha | x' \rangle}_{\psi_\alpha^*(x')} \left(-i\hbar \frac{\partial}{\partial x'}\right) \underbrace{\langle x' | \alpha \rangle}_{\psi_\alpha(x')}$$

$$= \hbar k$$

4> $\langle \hat{p}^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$

$$\Rightarrow \langle \Delta \hat{p}^2 \rangle = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = \frac{\hbar^2}{2d^2} + \hbar^2 k^2 - \hbar^2 k^2$$

$$= \frac{\hbar^2}{2d^2}$$

$$\Rightarrow \quad \langle (\Delta \hat{p})^2 \rangle \langle (\Delta \hat{x})^2 \rangle = \frac{\hbar^2}{2d^2} \cdot \frac{d^2}{2} = \frac{\hbar^2}{4}$$

Gaussian wave packet → minimum uncertainty.

↑

minimum uncertainty wavepaket.

$$\langle p' | \alpha \rangle = \phi_\alpha(p') = \frac{1}{\sqrt{2\pi\hbar}} \int dx'\, e^{-\frac{ip'x'}{\hbar}} \underbrace{\psi_\alpha(x')}_{\text{Gaussian wave}}$$

$$= \sqrt{\frac{d}{\hbar\sqrt{\pi}}}\, e^{-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}}$$

$$\psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}} e^{\left(ikx' - \frac{x'^2}{2d^2}\right)}$$

side note: if $d \to \infty$.

$\phi_\alpha(p') \to \delta(\tilde{p}')$

so. find particle at $\tilde{p}'$ in p'-space is becoming very certain. However. now. $\psi_\alpha(x')$ is extended everywhere in x'-space. finding particle in x'-space is becoming extremely uncertain.

---

*. 3-D

1> For momentum.

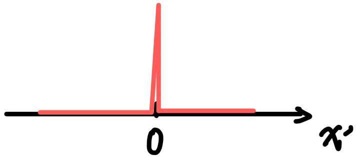
$$\hat{P}_{x'} = -i\hbar \frac{\partial}{\partial x'} \qquad \hat{P}_{y'} = -i\hbar \frac{\partial}{\partial y'} \qquad \hat{P}_{z'} = -i\hbar \frac{\partial}{\partial z'}$$

$$\hat{P}_{xyz} = -i\hbar \left(\vec{x} \frac{\partial}{\partial x'} + \vec{y} \frac{\partial}{\partial y'} + \vec{z} \frac{\partial}{\partial z'} \right)$$

$$= \left(-i\hbar \frac{\partial}{\partial x'}, -i\hbar \frac{\partial}{\partial y'}, -i\hbar \frac{\partial}{\partial z'} \right) \to \text{Gradient } \text{"}\nabla\text{"}$$

2> For δ - function.

1D. $\delta(x')$



$$\int dx' f(x')\,\delta(x') = f(0)$$

3D. $$\int d\overset{x',y',z'}{\overset{\uparrow}{X'}}\, f(X')\,\delta(X') = f(0,0,0)$$

$$\Rightarrow \delta(X') = \delta(x')\,\delta(y')\,\delta(z')$$

$$\Rightarrow \delta(X''-X') = \delta(x''-x')\,\delta(y''-y')\,\delta(z''-z')$$

$\Rightarrow$ for n-D space.

$$\delta(\underline{X}') = \delta(x_1')\,\delta(x_2')\,\delta(x_3')\cdots\delta(x_n')$$

5* Introduction to spin 1/2 system.

5-1. Stern-Gerlach Experiment.