on the other hand, if $[\hat{A}, \hat{B}] \neq 0$.

results from $\hat{C}$ depends on whether or not $\hat{B}$ measurement has actually been performed.

3.> Uncertainty :

$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$

*D14 operator $\Delta\hat{A} \equiv \hat{A} - \langle\hat{A}\rangle \mathbb{1}$

$\langle(\Delta\hat{A})^2\rangle$: dispersion of $\hat{A}$ ($\sim \sigma^2$)

$$= \langle(\hat{A} - \langle\hat{A}\rangle\mathbb{1})^2\rangle$$

$$= \langle (\hat{A})^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2\mathbb{1} \rangle$$

$$= \langle\hat{A}^2\rangle - \underline{2\langle\hat{A}\rangle\langle\hat{A}\rangle} + \underline{\langle\hat{A}\rangle^2}$$

$$= \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$$

variance or mean squared deviation.

(std deviation $\sigma = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$)

$\langle(\Delta\hat{A})^2\rangle_\psi$ vanish iff when state $|\psi\rangle$ in question. is an eigenstate of $\hat{A}$

Proof : $\hat{A}|\psi\rangle = a|\psi\rangle$

$$\rightarrow \boxed{(\hat{A} - \underbrace{\langle\hat{A}\rangle_\psi}_{\langle\psi|\hat{A}|\psi\rangle}\mathbb{1})}|\psi\rangle = a|\psi\rangle - \langle\psi|\underbrace{\hat{A}|\psi\rangle}_{a|\psi\rangle}\mathbb{1}|\psi\rangle$$

$\Delta\hat{A}$

$$= a|\psi\rangle - a|\psi\rangle = 0$$

uncertainty principle:

Given two Herm. operator $\hat{A}$ & $\hat{B}$, and a state $|\psi\rangle$ normalized.

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|$$

To proof. we need Lemmas (Intermediate theorems)

Lemma 1: Schwartz inequality:

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2.$$

$$|\vec{a}|^2|\vec{b}|^2 \geq |\underbrace{\vec{a}\cdot\vec{b}}_{|\vec{a}|\cdot|\vec{b}|\cdot\cos\theta}|^2$$

Lemma 2: The expectation value of $\hat{x}_H$ is real.

Lemma 3: The expectation value of Anti-Herm. is purely imaginary.

$$\hat{x}^+ = -\hat{x} \qquad \hat{x}|\psi\rangle = a|\psi\rangle$$

$$\rightarrow \langle\psi|\underbrace{\hat{x}|\psi\rangle}_{a|\psi\rangle} = a \underbrace{\qquad\qquad}_{a=-a^*} = a^*\overbrace{\langle\psi|\psi\rangle}^{1}$$

$$= \langle\psi|(-\hat{x}^+)|\psi\rangle = -\langle\hat{x}\psi|\psi\rangle = -\langle a\psi|\psi\rangle$$

w/ Lemma 1 Let $|\alpha\rangle = \Delta\hat{A}|\rangle$

may be any ket

$$|\beta\rangle = \Delta\hat{B}|\rangle$$

Based on $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2.$

$$\underbrace{\langle|\Delta\hat{A}\ \Delta\hat{A}|\rangle}_{\langle|\Delta\hat{A}^2|\rangle \to \langle(\Delta\hat{A})^2\rangle} \qquad \underbrace{\langle|\Delta\hat{B}\ \Delta\hat{B}|\rangle}_{\langle(\Delta\hat{B})^2\rangle} \qquad \boxed{|\langle|\Delta\hat{A}\ \Delta\hat{B}|\rangle|^2} \to \langle\Delta\hat{A}\,\Delta\hat{B}\rangle$$

$$\Delta\hat{A}\Delta\hat{B} = \underbrace{\frac{1}{2}[\Delta\hat{A},\Delta\hat{B}]}_{\frac{\Delta\hat{A}\Delta\hat{B}-\cancel{\Delta\hat{B}\Delta\hat{A}}}{2}} + \underbrace{\frac{1}{2}\{\Delta\hat{A},\Delta\hat{B}\}}_{\frac{\Delta\hat{A}\Delta\hat{B}+\cancel{\Delta\hat{B}\Delta\hat{A}}}{2}}$$

$$\begin{aligned}([\Delta\hat{A},\Delta\hat{B}])^\dagger &= (\Delta\hat{A}\Delta\hat{B}-\Delta\hat{B}\Delta\hat{A})^\dagger \\ &= (\Delta\hat{B}\Delta\hat{A}-\Delta\hat{A}\Delta\hat{B}) \\ &= -([\Delta\hat{A},\Delta\hat{B}])\end{aligned}$$

Previously given. $(\hat{X}\hat{Y})^\dagger = \hat{Y}^\dagger\hat{X}^\dagger$

$\Rightarrow [\Delta\hat{A},\Delta\hat{B}]$ is anti-Herm.

likewise $(\{\Delta\hat{A},\Delta\hat{B}\})$ is a Herm.

$$\to \langle\Delta\hat{A}\Delta\hat{B}\rangle = \underbrace{\frac{1}{2}\langle[\Delta\hat{A},\Delta\hat{B}]\rangle}_{\text{imaginary}} + \underbrace{\frac{1}{2}\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle}_{\text{real}}$$

$$\to |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2 = \frac{1}{4}|\langle[\Delta\hat{A},\Delta\hat{B}]\rangle|^2 + \underbrace{\frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2}_{\geq 0}$$

since $\underbrace{[\Delta\hat{A},\Delta\hat{B}]}_{\Delta\hat{A} = \hat{A}-\langle\hat{A}\rangle\mathbb{I}} = [\hat{A},\hat{B}]$ trivial proof.

$$\langle \psi|\Delta\hat{A}\,\Delta\hat{A}|\psi\rangle \langle \psi|\Delta\hat{B}\,\Delta\hat{B}|\psi\rangle \geq \boxed{|\langle \psi|\Delta\hat{A}\,\Delta\hat{B}|\psi\rangle|^2}$$

$$\Downarrow$$

$$\underline{\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle} \geq \boxed{\underbrace{\frac{1}{4}|\langle[\Delta\hat{A},\Delta\hat{B}]\rangle|^2}_{\geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2} + \underbrace{\frac{1}{4}|\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle|^2}_{\geq 0}}$$

$$\Rightarrow \boxed{\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq \frac{1}{4}|\langle[\hat{A},\hat{B}]\rangle|^2} \quad *$$

e.g. $\hat{A} = \hat{x}$, $\hat{B} = \hat{p}$ $\rightarrow$ $[\hat{x},\hat{p}] = i\hbar\mathbb{1}$

$$\Rightarrow \underbrace{\langle(\Delta\hat{x})^2\rangle}_{\delta_x^2}\underbrace{\langle(\Delta\hat{p})^2\rangle}_{\delta_p^2} \geq \underbrace{\frac{1}{4}|\langle i\hbar\mathbb{1}\rangle|^2}_{\frac{1}{4}\hbar^2}$$

$$\Rightarrow \delta_x^2\cdot\delta_p^2 \geq \frac{1}{4}\hbar^2 \Rightarrow \delta_x\delta_p \geq \frac{1}{2}\hbar$$

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3. Position, momentum & translation.

3-1 Continuous spectra

$\hat{x}$, & $\hat{p}$ are associated w/ continuous eigenvalue (spectra) e.g., $\hat{p}_z$ eigenvalue can

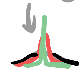
$\hat{W}|w'\rangle = w'|w'\rangle$ → continuous spectra

be anywhere $[-\infty, \infty]$

$g(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}}$

$\sigma \to 0 \to \delta(x)$

$\hat{A}|a'\rangle = a'|a'\rangle \longrightarrow$ discrete spectra.



$\delta(x)$: Dirac delta function

→ 0 anywhere, except "at" $x=0$

Analogy:

Discrete	continuous spectra
$\langle a'\|a''\rangle = \delta_{a'a''}$	$\langle w'\|w''\rangle = \delta(w'-w'')$
$\sum_{a'} \|a'\rangle\langle a'\| = \mathbb{1}$	$\int dw' \|w'\rangle\langle w'\| = \mathbb{1}$
$\|\alpha\rangle = \sum_{a'} \|a'\rangle\langle a'\|\alpha\rangle$	$\|\alpha\rangle = \int dw' \|w'\rangle\langle w'\|\alpha\rangle$
$\sum_{a'} \|\langle a'\|\alpha\rangle\|^2 = 1$	$\int dw' \|\langle w'\|\alpha\rangle\|^2 = 1$
$\langle a''\|\underbrace{\hat{A}\|a'\rangle}_{a'\|a'\rangle} = a'\delta_{a'a''}$	$\langle w''\|\hat{W}\|w'\rangle = w'\delta(w'-w'')$
if $a' \neq a'' \to = 0$; if $a' = a'' \to \neq 0$	if $w' \neq w'' \to = 0$; if $w' = w'' \to \neq 0$