$$\Rightarrow P = \frac{Y-\psi}{X-0} \Rightarrow \psi = Y - PX$$

$$\psi = Y(X) - P(X)\cdot X = Y(X(P)) - P(X(P))\cdot X(P)$$

---

conversely. if we know $\psi = \psi(P)$ $(= Y - PX)$

$$d\psi = dY - d(PX) = dY - \underbrace{P}_{\frac{\partial Y}{\partial x}}dx - x\,dP$$

$$= \cancel{dY} - \frac{\cancel{\partial Y}}{\partial x}\cancel{dx} - x\,dP = -x\,dP$$

$$\Rightarrow -x = \frac{d\psi}{dP} \rightarrow x = x(P) \rightarrow \boxed{P = P(x)}$$

eliminate "P"

$$Y = \psi - PX = \psi(P) - P\cdot X(P)$$

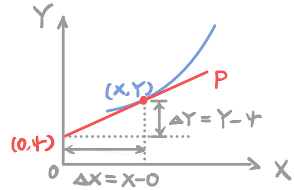
$$= \psi(P(X)) - P(X)\cdot X(P(X))$$

---

summary — single variable case.

$Y = Y(x)$	$\longleftrightarrow$	$\psi = \psi(P)$
$P = \frac{dY}{dx}$	$\longleftrightarrow$	$-x = \frac{d\psi}{dP}$
$\psi = -Px + Y$	$\longleftrightarrow$	$Y = \psi - (-x)\cdot P = xP + \psi$
Eliminate x & Y		Eliminate P & ψ
$\psi = \psi(P)$	$\longleftrightarrow$	$Y = Y(x)$

*. For multivariable case.



$$Y = Y(X_0, X_1, \cdots X_t)$$

$$\Rightarrow P_k \equiv \frac{\partial Y}{\partial X_k}$$

$$(\psi = Y - PX)$$

$$\underline{\psi = Y - \sum_k P_k X_k}$$

$$Y - \psi \cong dY$$

$$dY = \left(\frac{\partial Y}{\partial X_0}\right) dX_0 + \left(\frac{\partial Y}{\partial X_1}\right) dX_1 + \cdots$$

$$= P_0 dX_0 + P_1 dX_1 + \cdots$$

$$= \sum_k P_k \underbrace{dX_k}_{X_k - 0 = X_k} = \underbrace{\sum_k P_k X_k}_{} = \underbrace{Y - \psi}_{dY}$$

$$d\psi = \underline{dY} - \sum_k d(P_k X_k) = dY - \sum_k P_k dX_k - \sum_k X_k dP_k$$

$$= \cancel{\sum_k P_k dX_k} - \cancel{\sum_k P_k dX_k} - \sum_k X_k dP_k$$

$$= -\sum_k X_k dP_k$$

Now divide each side by dP_k

$$\rightarrow \frac{d\psi}{dP_k} \equiv \frac{-X_k dP_k + \left(-\sum_{j\neq k}^{t} X_j dP_j\right)}{dP_k} \quad \overset{\simeq}{\Rightarrow} -\sum_k X_k dP_k$$

$$= -X_k + \left(-\sum_{j\neq k} X_j \underbrace{\frac{\partial P_j}{\partial P_k}}_{0}\right)$$

$$\Rightarrow -X_k$$

Equivalence.

$$Y = Y(X_0, X_1, \cdots X_t) \longleftrightarrow \psi = \psi(P_0, P_1, \cdots P_t)$$

1> knowing $Y = Y(X_0, X_1, \cdots X_t)$

$$\rightarrow \psi = Y - \sum_k P_k X_k = Y(\cdots X_k \cdots) + \sum_k \underset{\substack{\Uparrow \\ \frac{\partial Y}{\partial X_k}}}{P_k}(X_k) \cdot X_k$$

2> since $P_k \equiv \dfrac{\partial Y}{\partial X_k} \rightarrow P_k = P_k(X_k)$

$$\Rightarrow \underbrace{X_k = X_k(P_k)}_{\text{Eliminate } X_k}$$

$$\Rightarrow \underline{\underline{\psi}} = Y(\cdots X_k(\underline{\underline{P_k}}) \cdots) + \sum_k P_k(X_k(\underline{\underline{P_k}})) \cdot X_k(\underline{\underline{P_k}})$$

likewise, if knowing $\psi = \psi(\cdots P_k \cdots) \Leftrightarrow Y = Y(\cdots X_k \cdots)$

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*. sub-space Legendre Transformation.

For multi-variable Legendre Transformation.

may be made w/ $(n+2)$ D of the Full space

$(t+2)$ D defined by $\underbrace{Y = Y(X_0, X_1 \cdots X_t)}_{(t+2)\,D}$. $(n < t)$

sub-space Legendre Transformed function

$$\underbrace{\underbrace{Y_\psi}_{1} = Y_\psi(\underbrace{P_0, P_1, \cdots P_n}_{n+1}}_{(n+2)\,D}, X_{n+1} \cdots X_t)$$

$$\Upsilon_\psi = \Upsilon_\psi(\{P_k\}, \{X_j\})$$
$$0 \leq k \leq n \qquad n+1 \leq j \leq t$$

$\widetilde{\partial\Upsilon}$ $\qquad$ $\widetilde{\partial\psi}$

1> $P_k \equiv \dfrac{\partial \Upsilon_\psi}{\partial X_k}$, $\quad -X_k \equiv \dfrac{\partial \Upsilon_\psi}{\partial P_k}$

$P_k = \dfrac{\partial \Upsilon}{\partial X_k} \qquad -X_k = \dfrac{\partial \psi}{\partial P_k}$

2> change of Υ_ψ (i.e. $d\Upsilon_\psi$) is contributed from.

$$\{P_k\} \quad \& \quad \{\cdots X_j \cdots\}$$
$$k \leq n \qquad n+1 \leq j \leq t$$

$$\Rightarrow d\Upsilon_\psi = d\Upsilon_\psi(\{P_k\}) + d\Upsilon_\psi(\{X_j\})$$
$$= \Sigma \underbrace{\frac{\partial \Upsilon_\psi}{\partial P_k}}_{-X_k} \cdot dP_k + \Sigma \underbrace{\frac{\partial \Upsilon_\psi}{\partial X_j}}_{P_j} dX_j$$
$$= -\sum_{k=0}^{n} X_k dP_k + \sum_{j=n+1}^{t} P_j dX_j$$

3> $d\Upsilon = \sum_0^t \left(\dfrac{\partial \Upsilon}{\partial X_i}\right) dX_i = \sum_0^t P_i dX_i$

$$= \sum_0^n P_k dX_k + \sum_{n+1}^{t} P_j dX_j$$

4> $d\Upsilon - d\Upsilon_\psi = \sum_0^n P_k dX_k + \cancel{\sum_{n+1}^{t} P_j dX_j} - \left(-\sum_{k=0}^{n} X_k dP_k + \cancel{\sum_{j=n+1}^{t} P_j dX_j}\right)$

$$= \sum_0^n P_k dX_k + \sum_{k=0}^{n} X_k dP_k$$
$$= \sum_0^n d(P_k X_k)$$

5> Integration

$$\int dY - \int dY_\psi = \int \sum_0^n d(P_k X_k)$$

$$\Rightarrow Y = Y_\psi + \sum_0^n P_k X_k$$

Lagrangian Mechanics.

Hamiltonian

2r # of variables.

$$Y \leftarrow \mathcal{L} = \mathcal{L}(\underbrace{v_1, v_2, \cdots v_r}_{\{v\}};\ \underbrace{q_1, q_2 \cdots q_r}_{\{q\}})$$

momenta $\equiv P_k \equiv \dfrac{\partial \mathcal{L}}{\partial v_k}$

(Generalized)

$$Y_\psi \leftarrow H = H(\underbrace{P_1, P_2, \cdots P_r}_{\{P\}};\ \underbrace{q_1, q_2, \cdots q_r}_{\{q\}})$$

$$-H = \mathcal{L} - \sum_1^r P_k v_k$$

$$\hat{H} = \hat{T} + \hat{V}$$

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3. Thermodynamic Potentials.

$Y = Y(X_0, X_1, \cdots X_t)$ $\quad$ $U = U(S, V, N_1, N_2 \cdots)$

$P_0, P_1, \cdots$ $\quad$ $T, -P, \mu$.

sub-space Legendre Transformation.

$Y_\psi = Y_\psi(P_0, X_1, \cdots X_t)$ $\quad$ $U_F = U_F(T, V, N_1 \cdots)$

* Helmholtz potential

$$F \equiv U[T]$$