6. Propagator & Feynman Path Integral.

6-1 Propagator.

* Def: Propagator — a function of two space-time points (i.e., a final position & time (x'', t), and an initial on (x', t_0))

$$K(x'', t; x', t_0) = \underbrace{\langle x'' | \hat{u}(t, t_0) | x' \rangle}$$

matrix elemen of time-evolution operator in the eigenbasis of position

$\downarrow$ Amplitude to find particle at (x'', t), given it was at (x', t_0)

$$\begin{aligned} |\alpha, t_0; t\rangle &= \hat{u}(t, t_0) |\alpha, t_0\rangle \\ &= e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} |\alpha, t_0\rangle \\ &= \sum_{a'} |a'\rangle\langle a'|\alpha, t_0\rangle\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)} \end{aligned}$$

$$\Rightarrow \psi_\alpha(x'', t) = \langle x'' | \alpha, t\rangle$$

$$= \sum_{a'} \underbrace{\langle x'' | a'\rangle}_{\psi_{a'}(x'')} \underbrace{\langle a' | \alpha, t_0\rangle}_{C_{a'}} e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}$$

$$C_{a'} = \int d^3x' \overbrace{\langle a' | x'\rangle}^{\psi^*_{a'}(x')} \overbrace{\langle x' | \alpha, t_0\rangle}^{\psi_\alpha(x', t_0)} = \int d^3x'\, \psi^*_{a'}(x')\, \psi_\alpha(x', t_0)$$

$$\underbrace{\int d^3x' \, |x'\rangle\langle x'|}_{\mathbb{1}}$$

$$\Rightarrow \underbrace{\psi_\alpha(x'',t)}_{\text{Final}} = \sum_{a'} \psi_{a'}(x'')\int d^3x'\, \psi^*_{a'}(x')\, \psi_\alpha(x',t_0)\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}$$

$$= \int d^3x' \boxed{\sum_{a'} \psi_{a'}(x'')\, \psi^*_{a'}(x')\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}}\ \underbrace{\psi_\alpha(x',t_0)}_{\text{initial}}$$

$$\sum_{a'} \psi_{a'}(x'')\, \psi^*_{a'}(x')\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}$$

$$= \sum_{a'} \langle x''|a'\rangle\langle a'|x'\rangle\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}$$

$$= \langle x''|\, e^{-\frac{i}{\hbar}E_{a'}(t-t_0)}\, |x'\rangle = \langle x''|\, \hat{u}\, |x'\rangle \Rightarrow K(x'',t;\, x',t_0)$$

matrix element of $\hat{u}$ in the position eigenbasis.

$$\Rightarrow \boxed{\psi_\alpha(x'',t) = \int d^3x'\, K(x'',t;\, x',t_0)\, \psi_\alpha(x',t_0)}$$

NOTE: This is not the Feynman path integral; this is the so-called propagation of wavefunc.

Intuitive Breakdown:

1> Initial state: At t_0, the particle is desribed by a wavefunction $\psi_\alpha(x',t_0)$, meaning there could be a distribution of amplitudes across different starting position x'.

2> Propagation: For each starting point, there's a propagation Amplitude $K(x'',t;\, x',t_0)$ to reach final point at x'' at t. This propagation amplitude encodes How much "support" that path gives to the particle arriving at "x''".

3> Summation (Integral): Total amplitude arriving at x'' is then a superposition of all possible contributions from each x' (i.e., each origin)

you weigh each point by: ① How likely it was to start at x' ($\psi_\alpha(x', t_0)$)

② How likely it is getting from x' to x''

($K(x'', t; x', t_0)$)

Properties of $K(x'', t; x', t_0)$

1> $\lim_{t \to t_0} K(x'', t; x', t_0) = K(x'', t_0; x', t_0)$

$$= \langle x'' | x' \rangle = \delta^3(x'' - x')$$

2> $\underline{i\hbar \frac{\partial}{\partial t} K(x'', t; x', t_0)} = \langle x'' | i\hbar \frac{\partial}{\partial t} e^{-\frac{i}{\hbar}\hat{H}(t-t_0)} | x' \rangle$

$$= \langle x'' | \hat{H} \underbrace{\hat{u} | x' \rangle}_{\text{" } |\alpha, t_0; t\rangle \text{"}}$$

Let's set $\hat{u} | x' \rangle = | \alpha, t_0; t \rangle$

$\Rightarrow \langle x'' | \hat{H} | \alpha, t_0; t \rangle$

$$= \left[-\frac{\hbar^2}{2m} \nabla''^2 + V(x'') \right] \langle x'' \underline{| \alpha, t_0; t \rangle}$$

$\hat{u}|x'\rangle$

$$= \left[-\frac{\hbar^2}{2m} \nabla''^2 + V(x'') \right] K(x'', t; x', t_0)$$

$\Rightarrow$ $K(x'', t; x', t_0)$ satisfies Schrödinger eq.

Understand K

① if we have a wavefunction initially given by

$$\psi(x, t_0) = \delta^3(x - x')$$

Highly Localized state at x', at t_0

Final time t. $\psi(x'', t) = K(x'', t; x', t_0)$

$$\Uparrow$$

$$\psi(x'', t) = \int d^3x \, K(x'', t; x', t_0) \underbrace{\psi(x, t_0)}_{\delta^3(x - x')}$$

only $x = x'$. survive

② $K(x'', t; x', t_0)$

Transition amplitude from space time (x', t_0) to (x'', t)

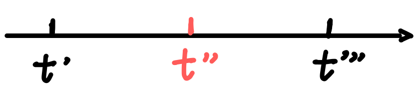
$$= \langle x'' | e^{-\frac{i}{\hbar}\hat{H}(t - t_0)} | x' \rangle$$

$$= \underbrace{\langle x'' | e^{-\frac{i}{\hbar}\hat{H}t}}_{\langle x'', t|} \cdot \underbrace{e^{\frac{i}{\hbar}\hat{H}t_0} | x' \rangle}_{|x', t_0\rangle} = \langle x'', t | x', t_0 \rangle$$

more generally $t_0 \doteq t'$, $t \doteq t''$

$\rightarrow$ Heisenberg picture position eigen basis.

6-2. Composition property:



$$\Rightarrow \langle x''', t''' | x', t' \rangle$$

$$= \int d^3x'' \langle x''', t''' | x'', t'' \rangle \langle x'', t'' | x', t' \rangle$$