$\hat{W}|w'\rangle = w'|w'\rangle$ be anywhere $[-\infty, \infty]$

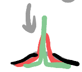
↓

continuous spectra

$g(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{\sigma^2}}$

$\sigma \to 0 \to \delta(x)$

$\hat{A}|a'\rangle = a'|a'\rangle \longrightarrow$ discrete spectra.



$\delta(x)$: Dirac delta function

→ 0 anywhere, except "at" $x=0$

Analogy:

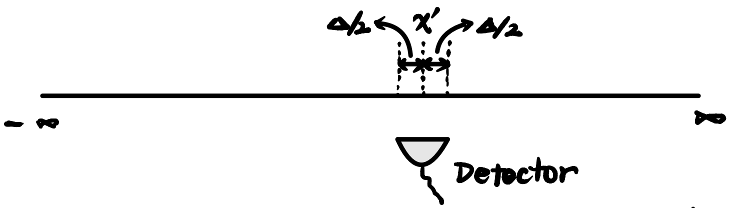
Discrete	continuous spectra
$\langle a'\|a''\rangle = \delta_{a'a''}$	$\langle w'\|w''\rangle = \delta(w'-w'')$
$\sum_{a'} \|a'\rangle\langle a'\| = \mathbb{1}$	$\int dw' \|w'\rangle\langle w'\| = \mathbb{1}$
$\|\alpha\rangle = \sum_{a'} \|a'\rangle\langle a'\|\alpha\rangle$	$\|\alpha\rangle = \int dw' \|w'\rangle\langle w'\|\alpha\rangle$
$\sum_{a'} \|\langle a'\|\alpha\rangle\|^2 = 1$	$\int dw' \|\langle w'\|\alpha\rangle\|^2 = 1$
$\langle a''\|\underbrace{\hat{A}\|a'\rangle}_{a'\|a'\rangle} = a'\delta_{a'a''}$	$\langle w''\|\hat{W}\|w'\rangle = w'\delta(w'-w'')$
$\begin{cases} \text{if } a' \neq a'' \to = 0 \\ \text{if } a' = a'' \to \neq 0 \end{cases}$	$\begin{cases} \text{if } w' \neq w'' \to = 0 \\ \text{if } w' = w'' \to \neq 0 \end{cases}$

3-2. Position eigenket & measurement.

$\hat{x}|x'\rangle = x'|x'\rangle$

↳ dimension of length e.g. [cm].

$$|\alpha\rangle = \underbrace{\int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|}_{\mathbb{1}}\alpha\rangle$$



locate particle in $(x'-\Delta/2,\ x'+\Delta/2)$

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx'|x'\rangle\langle x'|\alpha\rangle \xrightarrow{\text{measure}} \int_{x'-\Delta/2}^{x'+\Delta/2} dx''|x''\rangle\langle x''|\alpha\rangle$$

abrupt change.

if $\langle x''|\alpha\rangle = \text{const.}$ in $(x'-\Delta/2,\ x'+\Delta/2)$

$$\longrightarrow \boxed{P = |\langle x'|\alpha\rangle|^2 \underbrace{dx'}_{\Delta\ (\text{small})}} \longrightarrow \text{Probability density.}$$

$x' \pm \left(\frac{dx'}{2}\right) = \Delta/2$

$$P_{total} = \int_{-\infty}^{\infty} dx' |\langle x'|\alpha\rangle|^2 = 1 \qquad (\text{if } \langle\alpha|\alpha\rangle = 1)$$

$$\int_{-\infty}^{\infty} dx' \langle\alpha|x'\rangle\langle x'|\alpha\rangle$$

$$\underbrace{\int_{-\infty}^{\infty} dx' |x'\rangle\langle x'|}_{\mathbb{1}}$$

For 3D case: ket $|\mathbf{x}'\rangle = \underbrace{|x', y', z'\rangle}_{\text{simultaneous eigenket of } \hat{x},\ \hat{y},\ \hat{z}}$

capital

$\hat{x}|X'\rangle = x'|X'\rangle$, $\hat{y}|X'\rangle = y'|X'\rangle$, $\hat{z}|X'\rangle = z'|X'\rangle$

capital X

$\Rightarrow$ $[\hat{x}_i, \hat{x}_j] = 0$ $\hat{x}_1, \hat{x}_2, \hat{x}_3$ stand for $\hat{x}, \hat{y}, \hat{z}$

$\rightarrow$ $|\alpha\rangle = \int d^3x' |x'\rangle\langle x'|\alpha\rangle$

3-3. Translation operator.

$$\hat{T}(dx')|x'\rangle = |x'+dx'\rangle$$

$$\hat{T}(dx')|\alpha\rangle = \hat{T}(dx')\int d^3x' |x'\rangle\langle x'|\alpha\rangle$$

$$= \int d^3x' \underbrace{|x'+dx'\rangle}_{x'}\underbrace{\langle x'}_{x'-dx'}|\alpha\rangle$$

$$= \int d^3x' |x'\rangle\langle x'-dx'|\alpha\rangle$$

Properties of $\hat{T}(dx')$:

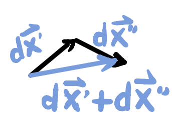
① if $\langle\alpha|\alpha\rangle = 1$ we require $\underbrace{\langle\alpha|\hat{T}^\dagger(dx')}_{\langle\alpha'|}\underbrace{\hat{T}|\alpha\rangle}_{|\alpha'\rangle} = 1$, $\boxed{\hat{T}^\dagger(dx')\hat{T}} \Rightarrow \mathbb{1}$

(i.e., probability is reserved)

$\Rightarrow$ $\underbrace{\hat{T}^\dagger(dx')\hat{T}(dx') = \mathbb{1}}_{\text{Unitary}}$.

② $\hat{T}(dx'')\hat{T}(dx') = \hat{T}(dx'+dx'')$

vector sum.



③ if $dx' \to 0$ $\quad \lim_{dx' \to 0} \hat{T}(dx') = \mathbb{1}$.

④ $\hat{T}(-dx') = \hat{T}^{-1}(dx')$ $\quad \Big(\underline{\hat{T}(-dx')} \cdot \underline{\hat{T}(dx')}$ $\nearrow \hat{T}^{-1}(dx')$

$= \hat{T}(-dx' + dx') = \hat{T}(0) = \mathbb{1} \Big)$

To have ①–④ satisfied:

$$\hat{T}(dx') = \mathbb{1} - i \hat{K} dx'$$

$\hat{K} \to$ Generator

Herm. $\hat{K}_x, \hat{K}_y, \hat{K}_z$

Validity:

① $$\hat{T}^{+}\hat{T} = (\mathbb{1} + i\hat{K}dx')(\mathbb{1} - i\hat{K}dx')$$

$$= \mathbb{1} - i\underbrace{(\hat{K} - \hat{K})dx'}_{=0} - \underbrace{(\cdots)(dx')^2}_{\approx 0 \ \ 2^{nd} \text{ order}}$$

$$= \mathbb{1}$$

② $\hat{T}(dx'')\hat{T}(dx') = \cdots$ (Practice after class)

$$= \mathbb{1} - i\hat{K}(dx' + dx'')$$

③ $dx' \to 0 \Rightarrow \hat{T}(dx') = \mathbb{1}$. (Trivial)

④ $$\hat{T}(-dx') \cdot \hat{T}(dx') = \hat{T}(-dx' + dx') = \mathbb{1}$$

$$\hat{T}(dx')\hat{T}(-dx') = \hat{T}(\cdots) = \mathbb{1}$$

$\Rightarrow$ inverse exists for $\hat{T}$

$$\Rightarrow \quad \hat{T}(-dx') = \hat{T}^{-1}(dx')$$

Fundamental properties of $\hat{x}$ & $\hat{k}$

$$\hat{x}\,\hat{T}(dx')|x'\rangle = \hat{x}|x'+dx'\rangle = (x'+dx')|x'+dx'\rangle$$

$$\hat{T}(dx')\,\hat{x}|x'\rangle = \hat{T}(dx')\,x'|x'\rangle = x'\underbrace{|x'+dx'\rangle}_{\approx |x'\rangle}$$

$$\Rightarrow \quad [\hat{x}, \hat{T}(dx')] = \mathbb{1}\,dx'$$

$$\Rightarrow \quad \hat{x}(\mathbb{1} - i\hat{K}dx') - (\mathbb{1} - i\hat{K}dx')\hat{x} = \overset{\cdot i}{-i\hat{x}\hat{K}\cancel{dx'}} + \overset{\cdot i}{i\hat{K}\hat{x}\cancel{dx'}}$$

$$= \underset{\cdot i}{\mathbb{1}\cancel{dx'}}$$

$$\Rightarrow \quad [\hat{x}, \hat{K}] = \mathbb{1}\cdot i$$

$$\Rightarrow \quad [\hat{x}_i, \hat{k}_j] = \mathbb{1}\cdot i\cdot\delta_{ij}$$

3-4. momentum (as translation generator)

How?

1> For spatial translation $\underline{\hat{T}(dx)\cdot f(x)} = f(x+dx)$

$$= \underline{f(x-dx)}$$

2> Expansion for infinitesimal shift:

$$\underline{f(x-dx)} = f(x) + \frac{df(x)}{dx}(-dx) + O(dx^2)$$

$$\approx \boxed{f(x) - \frac{df(x)}{dx}\cdot dx} \quad \Big\} \; \hat{T}(dx) = 1 - \frac{d}{dx}dx$$

$$\hat{T}(dx)f(x) = $$

3> General form of unitary operator:

$$\hat{U}(\lambda) = e^{-i\lambda\hat{G}}$$