For $|\alpha\rangle, |\beta\rangle$

$$\underbrace{\langle\beta|\alpha\rangle}_{} = \int dx' \underbrace{\langle\beta|x'\rangle}_{\psi_\beta(x')^*} \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')} = \int dx' \, \psi_\beta(x')^* \psi_\alpha(x')$$

→ Probability amplitude for state $|\alpha\rangle$ to be found in state $|\beta\rangle$

$$\langle\beta|\hat{A}|\alpha\rangle = \int dx' \int dx'' \underbrace{\langle\beta|x'\rangle}_{\psi_\beta^*(x')} \langle x'|\hat{A}|x''\rangle \underbrace{\langle x''|\alpha\rangle}_{\psi_\alpha(x'')}$$

$$= \int dx' \int dx'' \, \psi_\beta^*(x') \underbrace{\langle x'|\hat{A}|x''\rangle}_{\text{Matrix element}} \psi_\alpha(x'')$$

e.g., $\hat{A} = \hat{x}^2$

$$\rightarrow \langle x'|\hat{x}^2|x''\rangle = \langle x'|\underbrace{x''^2}_{\text{scalar}}|x''\rangle = x''^2 \langle x'|x''\rangle$$

$$= x''^2 \, \delta(x'-x'')$$

$$\Rightarrow \langle\beta|\hat{A}|\alpha\rangle = \underline{\int dx' \, \psi_\beta^*(x') \, x'^2 \, \psi_\alpha(x')}$$

has to be reduced to single integral (otherwise, Dirac delta will quench everything)

4-2. Momentum space operator

Goal : $\langle x'|\hat{p}|\alpha\rangle = ?$

Previously $\left\{ \boxed{\hat{K} = -i\hbar \frac{d}{dx}} \right.$

$\Downarrow$

$\hat{P}$

$$\hat{\mathcal{T}}(\Delta x')|\alpha\rangle = \left(1 - \frac{i\hat{p}\Delta x'}{\hbar}\right)|\alpha\rangle$$

$$= \int dx'\, \hat{\mathcal{T}}(\Delta x')\, |x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx'\, |x'+\Delta x'\rangle\langle x'|\alpha\rangle$$

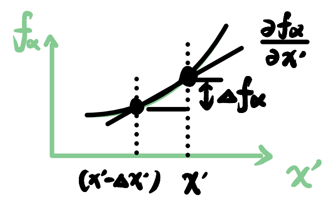
$$= \int dx'\, |x'\rangle \underbrace{\langle x'-\Delta x'|\alpha\rangle}$$

$x'-\Delta x'$
$\Uparrow$
$f_\alpha(\tilde{x}')$
$\downarrow$
y_α

$$= \int dx'\, |x'\rangle \Big(\underbrace{\langle x'|\alpha\rangle}_{f_\alpha(x')} - \underbrace{\left(\frac{\partial}{\partial x'}\langle x'|\alpha\rangle \Delta x'\right)}_{\Delta f_\alpha(x')} \Big)$$

$$\Rightarrow \underbrace{\int dx'\, \cancel{|x'\rangle}\langle x'|\alpha\rangle}_{|\alpha\rangle} - \int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\alpha\rangle \cancel{\Delta x'}$$

$$= \left(\cancel{1} - \frac{i\hat{p}\cancel{\Delta x'}}{\hbar}\right)\cancel{|\alpha\rangle}$$



$$\Rightarrow \hat{p}|\alpha\rangle = \int dx'\, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$\Rightarrow \langle x'|\hat{p}|\alpha\rangle = \int dx' \langle x'|x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$\Rightarrow \langle x'|\hat{p}|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

also $\Rightarrow \langle x'|\hat{p}^n|\alpha\rangle = (-i\hbar)^n \dfrac{\partial^n}{\partial x'^n}\langle x'|\alpha\rangle$

$\Rightarrow$ matrix element of $\hat{p}$ in the x'-representation.

set $|\alpha\rangle = |x''\rangle \Rightarrow \langle x'|\hat{p}|x''\rangle = -i\hbar \dfrac{\partial}{\partial x'}\langle x'|x''\rangle$

$$= -i\hbar \frac{\partial}{\partial x'}\delta(x'-x'')$$

$$\Rightarrow \langle\beta|\hat{p}|\alpha\rangle = \int dx'' \int dx' \langle\beta|x''\rangle \underbrace{\langle x''|\hat{p}|x'\rangle}_{\text{Matrix element}} \langle x'|\alpha\rangle$$

$$= \int dx' \, \psi_\beta^*(x') \, \hat{p} \, \psi_\alpha(x')$$

$$\Rightarrow \langle\beta|\hat{p}^n|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \, (-i\hbar)^n \left(\frac{\partial^n}{\partial x'^n}\right) \psi_\alpha(x')$$

4-3. momentum space wavefunction.

$$\hat{x}\,|x'\rangle = x'\,|x'\rangle$$

$$\rightarrow \quad \hat{p}\,|p'\rangle = p'\,|p'\rangle$$

$$\langle x'|x''\rangle = \delta(x'-x'')$$

$$\rightarrow \quad \langle p'|p''\rangle = \delta(p'-p'')$$

$$|\alpha\rangle = \int dp' \, |p'\rangle \underbrace{\langle p'|\alpha\rangle}_{\phi_\alpha(p')} \qquad \underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$

$|\langle p'|\alpha\rangle|^2 dp'$ is the probality of finding the particle w/ p' in the interval of dp'

if $|\alpha\rangle$ is normed. $\langle\alpha|\alpha\rangle = 1$

$$= \int dp' \langle\alpha|p'\rangle\langle p'|\alpha\rangle$$

$$= \int dp' \, \phi_\alpha^*(p') \, \phi_\alpha(p')$$

$$= \int dp' \, |\phi_\alpha(p')|^2 = 1$$

* The connection between x- & p- representation.

recall transformation matrix ($\hat{U}$)

$$|b_i\rangle = \hat{U}|a_i\rangle \Rightarrow \langle a_j|\hat{U}|a_i\rangle = \boxed{\langle a_j|b_i\rangle}$$

matrix element of $\hat{U}$

old

new eigenbasis.

$\Rightarrow \langle x'|p'\rangle$ matrix element transforming x'- to p'- representation.

$$\Rightarrow \underbrace{\langle x'|\hat{p}|p'\rangle}_{\langle x'|p'|p'\rangle} = -i\hbar\frac{\partial}{\partial x'}\langle x'|p'\rangle$$

$$p'\underbrace{\langle x'|p'\rangle}_{f_p(x')} = -i\hbar\frac{\partial}{\partial x'}\underbrace{\langle x'|p'\rangle}_{f_p(x')}$$

$$\Rightarrow p'f = -i\hbar\frac{\partial f}{\partial x'} \Rightarrow \int\frac{ip'}{\hbar}dx' = \int\frac{df}{f}$$

$$\Rightarrow f = \langle x'|p'\rangle = \boxed{N}\, e^{\frac{ip'x'}{\hbar}}$$

const.

Plane wave.

$e^{ix} = \cos x + i\sin x$

Probability amplitude of momentum eigenstate specified by p' to be found at x'

For N: $\underbrace{\langle x'|x''\rangle}_{\delta(\cancel{x'-x''})} = \int dp' \underbrace{\langle x'|p'\rangle}_{Ne^{\frac{ip'x'}{\hbar}}}\underbrace{\langle p'|x''\rangle}_{N^* e^{-\frac{ip'x''}{\hbar}}}$

$$= \int dp' |N|^2 e^{ip'(x'-x'')/\hbar}$$

$$= |N|^2 \cdot 2\pi \cdot \hbar \cdot \delta(\cancel{x'-x''})$$

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-\alpha)}dk = \delta(x-\alpha)\right)$$

$\Rightarrow N = \frac{1}{\sqrt{2\pi\hbar}}$ $\Rightarrow \langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x'/\hbar}$

*. How position-space wavefunction is related to momentum-space wavefunction:

$$\langle x'|\alpha\rangle = \psi_\alpha(x') = \int dp' \underbrace{\langle x'|p'\rangle}_{\frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ip'x'}{\hbar}}}\underbrace{\langle p'|\alpha\rangle}_{\phi_\alpha(p')}$$

$$\langle p'|\alpha\rangle = \phi_\alpha(p') = \int dx' \underbrace{\langle p'|x'\rangle}_{\underbrace{\langle x'|p'\rangle^*}_{\downarrow\ \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-ip'x'}{\hbar}}}}\underbrace{\langle x'|\alpha\rangle}_{\psi_\alpha(x')}$$