$\Rightarrow$ i.e. the amplitude for a particle to go from (x', t') to (x''', t''') is sum over all possible intermediate positions x'' at t''

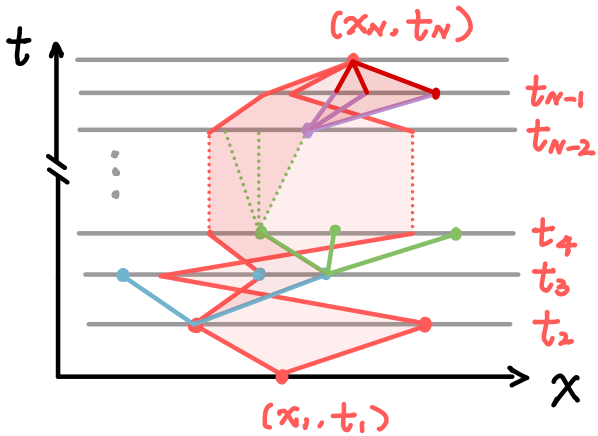
6-3 Path Integral

consider transition from (x_1, t_1) to (x_N, t_N)

$$t_j - t_{j-1} = \Delta t = \frac{t_N - t_1}{N-1}$$

$$\Rightarrow \langle x_N, t_N | x_1, t_1 \rangle$$

$$= \int dx_{N-1} \int dx_{N-2} \cdots \int dx_2 \underbrace{\langle x_N, t_N | x_{N-1}, t_{N-1} \rangle}_{K(x_N, t_N; x_{N-1}, t_{N-1})} \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_2, t_2 | x_1, t_1 \rangle$$



$$\Rightarrow K(x_N, t_N; x_1, t_1) = \lim_{N\to\infty} \int dx_2 \int dx_3 \cdots \int dx_{N-1} \prod_{j=2}^{N} K(x_j, t_j; x_{j-1}, t_{j-1})$$

path integral is constructed by composing infinetesimally small segment of propagators over time steps.

6-4 A more precise formulation

(Feynman Path Integral Formulation for QM)

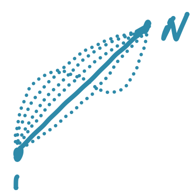
1. Classic Mechanics based on Lagrangian function.

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - V(x)$$

$x = x(t), \ \dot{x} = \dot{x}(t)$

Hamilton's principle (principle of stationary action)

⇒ Given a fixed starting and ending space-time point (x_1, t_1) & (x_N, t_N). Nature will choose the path, so that the action is stationary (typically, the action is minized)



Now. def. — action: action S is defined as

$$S(N, 1) = \int_{t_1}^{t_N} dt \, \mathcal{L}(x, \dot{x}, t)$$

⇒ S is sort of the total "effort" nature has put

⇒ if S is stationary ⇒ $\delta S = 0$

$$\Rightarrow \quad \delta \int_{t_1}^{t_N} dt\, L(x, \dot{x}, t) = 0$$

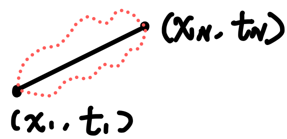
typically, S is minized.

$\Rightarrow$ Nature is "effecient", not wasting energy.

$\Rightarrow$ NOTE: Newton's Law is just a special case.

& can be derived from this (Hamilton's principle)

e.g., for a free particle, i.e., $V(x)=0$



* $\Rightarrow$ $\boxed{\text{classical mechanics is contrasting QM, in the way that the path is definite in space-time plane, while in QM, All possible paths play roles.}}$

$\Rightarrow$ Question: How to reconcile classical mechanics w/ QM in a smooth manner, in the limit of $\hbar \to 0$

(recall Ehrenfest's Theorem, likewise, this is a reasonable question)

2> Feynman's Question based on Dirac's NOTE.

* $\boxed{e^{i\int_{t_1}^{t_2} \frac{dt\, L(x,\dot{x},t)}{\hbar}} \quad \boxed{\text{corresponds to}}^{?} \quad \langle x_2, t_2 | x_1, t_1 \rangle}$

seed for Feynman's path integral and much of modern Quantum Theory.

what did P. Dirac mean above?

Now consider: $\underline{S(n, n-1)} = \int_{t_{n-1}}^{t_n} dt\, L(x, \dot{x}, t)$ ← classical mechanics.

$S(n, n-1)$ → small segment (prescribed)

⇒ then we're instructed by Dirac to associate this segment with $e^{\frac{i}{\hbar} S(n, n-1)}$

Now, from $1 \rightarrow N \Leftrightarrow \prod_{n=2}^{N} e^{\frac{i}{\hbar} S(n, n-1)}$

$$= e^{\frac{i}{\hbar} \underbrace{\sum_{n=2}^{N} S(n, n-1)}_{\text{total action from } 1 \rightarrow N}}$$

$$= e^{\frac{i}{\hbar} S(N, 1)}$$

NOTE. This is NOT $\langle x_N, t_N | x_1, t_1 \rangle$ yet.

*. This is just one particular path contributing to it.

⇒ then, the total contribution:

$$\langle x_N, t_N | x_1, t_1 \rangle \underset{?}{\sim} \sum_{\text{All paths}} e^{\frac{i}{\hbar} S(N,1)}$$

Now, if $\hbar \to 0$

⇒ small change in S → big change in $e^{\frac{i}{\hbar}S}$

⇒ $e^{i\frac{S}{\hbar}}$ will oscillate violently.

⇒ most of paths do not contribute anymore

Due to interference.

to the occurance of $x_1, t_1 \to x_N, t_N$

⇒ Except for the one: $\delta S(N,1) = 0$