$\Rightarrow$ Physical significance of $\hat{a}^+$ & $\hat{a}$:

$$\hat{N}(\hat{a}^+|\lambda\rangle) = ([\hat{N},\hat{a}^+] + \hat{a}^+\hat{N})|\lambda\rangle$$

$$= \hat{a}^+|\lambda\rangle + \lambda\hat{a}^+|\lambda\rangle$$

$$= (\lambda+1)(\hat{a}^+|\lambda\rangle) \quad C_+|\lambda+1\rangle$$

likewise

$$\hat{N}(\hat{a}|\lambda\rangle) = ([\hat{N}.\hat{a}] + \hat{a}\hat{N})|\lambda\rangle$$

$$= -\hat{a}|\lambda\rangle + \lambda\hat{a}|\lambda\rangle$$

$$= (\lambda-1)(\hat{a}|\lambda\rangle) \quad C_-|\lambda-1\rangle$$

$\Rightarrow$ $\hat{a}^+|\lambda\rangle = C_+|\lambda+1\rangle$ $\Rightarrow$ " creation/Rising "

$\hat{a}|\lambda\rangle = C_-|\lambda-1\rangle$ $\Rightarrow$ " Annihilation/Lowering"

$$\Rightarrow \quad \langle\lambda|\hat{a}^+\hat{a}|\lambda\rangle = \langle\lambda|\hat{N}|\lambda\rangle = \lambda\underbrace{\langle\lambda|\lambda\rangle}_{1} = \lambda$$

$$\langle\lambda-1|C_-^* \; C_-|\lambda-1\rangle$$

$$|C_-|^2$$

$\Rightarrow$ 1> $\lambda \geq 0$

2> $C_- = \sqrt{\lambda}$ $(C_- \in \mathbb{R})$

likewise $\Rightarrow$ $C_+ = \sqrt{\lambda+1}$

*.

$$\Rightarrow \quad \hat{a}|\lambda\rangle = \sqrt{\lambda}\,|\lambda-1\rangle, \quad \hat{a}^2|\lambda\rangle = \sqrt{\lambda(\lambda-1)}\,|\lambda-2\rangle$$

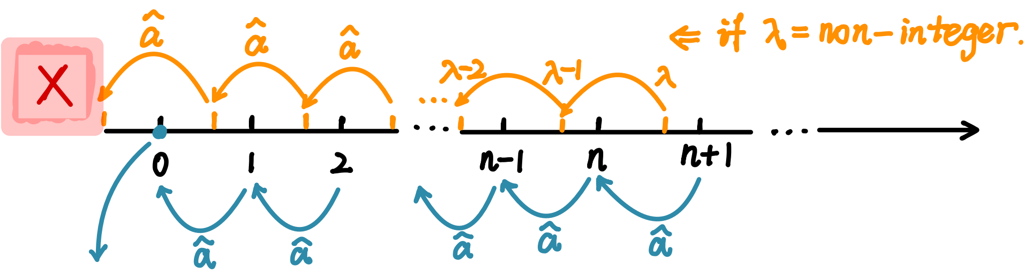
$$\cdots \quad \hat{a}^m|\lambda\rangle = \sqrt{\lambda(\lambda-1)\cdots(\lambda-m+1)}\,|\lambda-m\rangle$$

$$\hat{a}^+|\lambda\rangle = \sqrt{\lambda+1}\,|\lambda+1\rangle, \quad (\hat{a}^+)^2|\lambda\rangle = \sqrt{(\lambda+1)(\lambda+2)}\,|\lambda+2\rangle$$

$$\cdots \quad (\hat{a}^+)^m|\lambda\rangle = \sqrt{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}\,|\lambda+m\rangle$$

*. NOTE: 1> $\lambda = \langle\lambda|\hat{N}|\lambda\rangle \geq 0$

& 2> we postulate the existence of lowest vacuum state $|0\rangle$, such that $\hat{a}|0\rangle = 0$



$\hat{a}|0\rangle = 0$

$\Rightarrow$ some condition has to be met to reach $\lambda = 0$

$\hat{a}|\lambda\rangle = \sqrt{\lambda}\,(\quad) = 0$, if $\lambda = 0 \Rightarrow \lambda \in \mathbb{N}$

$\Rightarrow$ eigenvalue of $\hat{N}$ must be quantized / Numbered.

$\lambda \doteq n = 0, 1, 2, \cdots \qquad \hat{N}|n\rangle = n|n\rangle$

$\Rightarrow$ Energy (eigenvalue) is also quantized.

$$\rightarrow E_n = \hbar\omega(n + 1/2)$$

$\rightarrow (E_n)_{min} = E_0 = \hbar\omega/2 \Rightarrow$ (zero-point energy)

$\downarrow$

$|n\rangle = |0\rangle$

For $|0\rangle$

$\Rightarrow |1\rangle = \hat{a}^+|0\rangle$

$|2\rangle = \frac{\hat{a}^+}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^+)^2}{\sqrt{2}}|0\rangle$

$|3\rangle = \frac{\hat{a}^+}{\sqrt{3}}|2\rangle = \frac{(\hat{a}^+)^3}{\sqrt{3!}}|0\rangle$

$\vdots$

$$\boxed{|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle} \; *$$

$\Downarrow$

*. simultaneous eigenket of $\hat{N}$ & $\hat{H}$ (of QHO)

* $|n\rangle$ is also so-called Fock state. once you quantized a field. each mode behaves like an independent QHO. so you can describe each mode by $|n\rangle$ —— this is the birth of 2nd Quantization and Fock space.

$\Rightarrow$ Matrix element

$\hat{a}$: $\langle n'|\hat{a}|n\rangle = \langle n'|\sqrt{n}|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$

$\hat{a}^+$: $\langle n'|\hat{a}^+|n\rangle = \langle n'|\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\,\delta_{n',n+1}$

since $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+)$

$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^+ - \hat{a})$

$$\Rightarrow \langle n'|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

$$\langle n'|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right)$$

Energy eigenfunction is position space : $\langle x'|n\rangle$

Ground state of QHO

$$\langle x'|\hat{a}|0\rangle = 0 = \langle x'|\left[\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\right]|0\rangle$$

$$\langle x'|\hat{p}|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$\Rightarrow \sqrt{\frac{m\omega}{2\hbar}}\langle x'|\left(\hat{x} + \left(-i\hbar\frac{\partial}{\partial x'}\right)\frac{i}{m\omega}\right)|0\rangle = 0$$

$$\Rightarrow \langle x'|\left(\hat{x} + x_0^2\frac{d}{dx'}\right)|0\rangle$$

$$\downarrow$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\langle x'|\hat{x}|0\rangle = x'\underline{\langle x'|0\rangle} \rightarrow \psi_0(x')$$

$$\rightarrow \sum\langle x'|\hat{x}|x''\rangle\langle x''|0\rangle$$

$$= \sum x''\langle x'|x''\rangle\langle x''|0\rangle = \sum x''\,\delta_{x',x''}\langle x''|0\rangle$$

$$= x'\langle x'|0\rangle$$

$$\Rightarrow \langle x'|\hat{a}|0\rangle = \left(x' + x_0^2\frac{d}{dx'}\right)\underbrace{\langle x'|0\rangle}_{\psi_0(x')} = 0$$

$$\Rightarrow \boxed{\psi_0(x') = \frac{1}{\pi^{1/4}\sqrt{x_0}}\,e^{-\frac{1}{2}\left(\frac{x'}{x_0}\right)^2}}$$