3> General form of unitary operator:

$$\hat{U}(\lambda) = e^{-i\lambda \frac{\hat{G}}{\hbar}}$$

General.

$\Rightarrow$ it involves an arbitrary operator $\hat{G}$

$\Rightarrow$ it preserves the probability of a state.

$\hat{U}^{+}\hat{U} = \mathbb{1}$

$$e^{-i\lambda\hat{G}} = 1 + (-i\lambda\hat{G}) + \cdots$$

knowing $\hat{T}(dx) = \mathbb{1} - i\hat{k}dx$

$\Rightarrow \hat{T}(dx) = e^{-idx\hat{k}} = e^{-idx\hat{k}'/\hbar} \quad (\hbar\hat{k} = \hat{k}')$

$\Rightarrow \hat{T}(dx) = 1 - i\frac{dx}{\hbar}\hat{k}' + \mathcal{O}(dx^2)$

$\simeq 1 - i\frac{dx}{\hbar}\hat{k}'$

comparing: $\hat{T}(dx) \simeq 1 - \frac{d}{dx}dx$

$\Rightarrow 1 - \frac{d}{dx}dx = 1 - i\frac{dx}{\hbar}\hat{k}' \Rightarrow \boxed{\hat{k}' = -i\hbar\frac{d}{dx}}$

$\Downarrow$

$\hat{P}$

$\downarrow$

Dimension analysis.

$\hbar$: $[J\cdot S] = [J\cdot Hz^{-1}]$

Energy per Hz.

$\Rightarrow \hat{T}(dx) = \mathbb{1} - i\hat{k}dx = \mathbb{1} - i\frac{\hat{k}'}{\hbar}dx$

$\Rightarrow \hat{T}(dx) = \mathbb{1} - i\frac{\hat{P}}{\hbar}dx$

$\Rightarrow \hat{P} = \hbar\hat{K}$

$\hbar/dx$: $\underline{[J\cdot s\cdot cm^{-1}]}$

↳ $\underbrace{\text{unit for momentum.}}$

unit for $\hat{k}$ is $\dfrac{[J\cdot s\cdot cm^{-1}]}{[J\cdot s]} = cm^{-1}$

$$\Rightarrow \hat{T}(dx') = \mathbb{1} - \frac{i}{\hbar}\hat{p}\,dx'$$

$$\Rightarrow \underline{[\hat{x}_i, \hat{p}_j]} = \hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i = \hbar\hat{x}_i\hat{k}_j - \hbar\hat{k}_j\hat{x}_i$$

$$= \hbar[\hat{x}_i, \hat{k}_j] = \underline{\mathbb{1}\cdot i\cdot\hbar\cdot\delta_{ij}}$$

⇒ Position momentum uncertainty:

$$\langle(\Delta X)^2\rangle\langle(\Delta p)^2\rangle \geq \frac{\hbar^2}{4}$$

Now translate over $\Delta x'$

$$dx' = \lim_{N\to\infty}\frac{\Delta x'}{N}$$

$$\to \hat{T}(\Delta x'\cdot\underset{\substack{\downarrow\\ \text{unit vector.}}}{\vec{x}}) = \lim_{N\to\infty}\left(\mathbb{1} - \frac{i}{\hbar}\hat{p}\cdot\frac{\Delta x'}{N}\right)^N$$

$$= \lim_{N\to\infty}\left(\mathbb{1} + \frac{\overbrace{-(i/\hbar)\hat{p}\cdot\Delta x'}^{\text{"}x\text{"}}}{N}\right)^N$$

$$\boxed{= \lim_{N\to\infty}\left(\mathbb{1} + \frac{x}{N}\right)^N}$$

$$= e^x$$

$$= e^{-\frac{i}{\hbar}\hat{p}\cdot\Delta x'}$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \underline{e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}$$

$$\ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right)$$

$$\lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) \simeq n\cdot\frac{x}{n} = x$$

$$\Rightarrow e^{\lim_{n\to\infty}\ln(1+\frac{x}{n})^n} = e^x \Rightarrow e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$\begin{tikzcd} D \arrow[r, "x"] & B \\ A \arrow[u, "y"] \arrow[r, "x"'] & C \arrow[u, "y"'] \end{tikzcd}$$

$$\Rightarrow \hat{T}(\Delta y' \cdot \vec{y}) \cdot \hat{T}(\Delta x' \cdot \vec{x})$$

$$= \hat{T}(\Delta x' \cdot \vec{x}) \cdot \hat{T}(\Delta y' \cdot \vec{y})$$

$$\Rightarrow [\hat{T}(\Delta y' \cdot \vec{y}), \hat{T}(\Delta x' \cdot \vec{x})] = 0$$

$$= \left[\left(\boxed{\mathbb{1} - \frac{i \hat{P}_y \Delta y'}{\hbar}} - \frac{\hat{P}_y^2 (\Delta y')^2}{2\hbar^2} \cdots\right), \left(\boxed{\mathbb{1} - \frac{i \hat{P}_y \Delta y'}{\hbar}} - \frac{\hat{P}_y^2 (\Delta y')^2}{2\hbar^2} \cdots\right)\right]$$

$$\Leftarrow - \frac{(\Delta x')(\Delta y')[\hat{P}_y, \hat{P}_x]}{\hbar^2} = 0$$

$$\Rightarrow [\hat{P}_y, \hat{P}_x] = 0$$

$\Rightarrow$ Generally. $[\hat{P}_i, \hat{P}_j] = 0$ *.

$\Rightarrow \{\hat{P}_i\}$ form an Abelian Group: Group describes a system. where observables commute.

*. since. $[\hat{T}, \hat{P}] = 0$ and. $\hat{P}|P_i\rangle = P_i|P_i\rangle$

$\Rightarrow |P_i\rangle$ is eigenket of $\hat{T}$

3-5. canonical commutation:

Fundamental quantum behavior of conjugate variables

$$[\hat{x}_i, \hat{x}_j] = 0$$

$$[\hat{P}_i, \hat{P}_j] = 0$$

$$[\hat{x}_i, \hat{P}_j] = i\hbar \cdot \delta_{ij} \mathbb{1}.$$

4. wavefunctions in position. & momentum space.

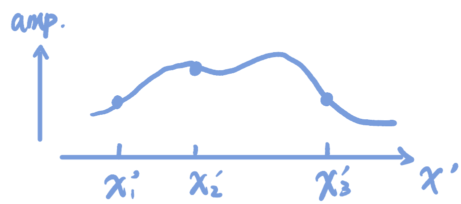
4-1. wavefunctions in position space.

$$\hat{x}|x'\rangle = x'|x'\rangle \qquad \underset{\uparrow}{\neq 0} \text{ iff } x''=x'$$

orthogonality: $\langle x''|x'\rangle = \delta(x''-x')$

$$|\alpha\rangle = \underbrace{\int dx'|x'\rangle\langle x'|}_{1}\boxed{\langle x'|\alpha\rangle}$$

$\downarrow$ $f_\alpha(x')$



a smooth function. $\psi_\alpha(x')$, in the 1D space.

(the points in the 1D. are labeled by x'), in this space. $\psi_\alpha(x')$ is a smooth function of x'.

and is called a wavefunction in x' space.

$\Rightarrow$ $\psi_\alpha(x') = \langle x'|\alpha\rangle$

$\hookrightarrow$ Probability amplitude for finding a particle in state $|x'\rangle$

consistent w/ postulate

$$P = |\langle x'|\alpha\rangle|^2 = |\psi_\alpha(x')|^2 = \psi_\alpha^*(x')\psi_\alpha(x')$$