1-2 t-dependent schrödinger equation

$$\hat{u}(t+dt,t_0) = \underline{\hat{u}(t+dt,t)}\,\hat{u}(t,t_0)$$

$$= \left(\mathbb{1} - i\frac{\hat{H}}{\hbar}dt\right)\underline{\hat{u}(t,t_0)}$$

$$\Rightarrow \underbrace{\hat{u}(t+dt,t_0) - \hat{u}(t,t_0)}_{d\hat{u}(t,t_0)} = \left(-i\frac{\hat{H}}{\hbar}dt\right)\hat{u}(t,t_0)$$

$$\Rightarrow \frac{\partial \hat{u}(t,t_0)}{\partial t} = \left(-i\frac{\hat{H}}{\hbar}\right)\hat{u}(t,t_0)$$

$$\Rightarrow \boxed{i\hbar\frac{\partial \hat{u}(t,t_0)}{\partial t} = \hat{H}\,\hat{u}(t,t_0)} \quad *$$

put $|\alpha,t_0\rangle$ on both L.H.S. & R.H.S.

$$\Rightarrow i\hbar\frac{\partial \hat{u}(t,t_0)}{\partial t}|\alpha,t_0\rangle = \hat{H}\,\underbrace{\hat{u}(t,t_0)|\alpha,t_0\rangle}_{|\alpha,t_0;t\rangle}$$

$$\Rightarrow \boxed{i\hbar\frac{\partial}{\partial t}|\alpha,t_0;t\rangle = \hat{H}\,|\alpha,t_0;t\rangle} \quad *$$

t-dependent schrödinger eq.

*. How to know the physical state at arbitrary time (t)?

Instead of solving $\boxed{i\hbar\frac{\partial}{\partial t}|\alpha,t_0;t\rangle = \hat{H}\,|\alpha,t_0;t\rangle}$

We solve $\boxed{i\hbar \frac{\partial \hat{U}(t,t_0)}{\partial t} = \hat{H}\hat{U}(t,t_0)}$

Because, $\hat{U}(t,t_0)|\alpha;t_0\rangle = |\alpha,t_0;t\rangle$

$$\boxed{\hat{U}(t_0+dt,t_0) = \mathbb{1} - i\frac{\hat{H}}{\hbar}dt}\ *$$

Based on nature of $\hat{H}$ $\begin{cases} \hat{H} = \hat{H} \\ \hat{H} = \hat{H}(t') \end{cases}$

1> if $\hat{H}$ is t-independent

$$\hat{U}(\underset{\downarrow \atop t_0+\Delta t}{t},t_0) = \lim_{N\to\infty} \hat{U}\Big(t_0+N\underbrace{\boxed{\frac{\Delta t}{N}}}_{dt},t_0\Big)$$

$$= \lim_{N\to\infty} \hat{U}(t_0+Ndt,t_0)$$

$$= \lim_{N\to\infty} \Big[\hat{U}(t_0+Ndt,t_0+(N-1)dt)$$
$$\cdot \hat{U}(t_0+(N-1)dt,t_0+(N-2)dt)$$
$$\vdots$$
$$\cdot \hat{U}(t_0+dt,t_0)\Big]$$

$$= \lim_{N\to\infty} \Big(\mathbb{1} - i\frac{\hat{H}}{\hbar}\underbrace{dt}_{\frac{\Delta t}{N} = \frac{t-t_0}{N}}\Big)^N$$

$$= \lim_{N\to\infty} \Big(\mathbb{1} - i\frac{\hat{H}}{\hbar}\cdot\frac{t-t_0}{N}\Big)^N$$

We set $x = -i\frac{\hat{H}}{\hbar}\cdot(t-t_0)$

$$\Rightarrow \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x = e^{-i\frac{\hat{H}}{\hbar}(t-t_0)}$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

similarly. $\Rightarrow \displaystyle\sum_{n=0}^{\infty}\frac{\left(-i\frac{\hat{H}}{\hbar}(t-t_0)\right)^n}{n!}$ ← Expansion Form.

2> if $\hat{H}$ is t-dependent. $\hat{H}=\hat{H}(t')$

$$\boxed{i\hbar\frac{\partial\hat{u}(t,t_0)}{\partial t} = \hat{H}\,\hat{u}(t,t_0)}$$

$$\hat{H} \rightarrow \hat{H}(t')$$

$$\Rightarrow \frac{d\hat{u}(t',t_0)}{\hat{u}(t',t_0)} = -\frac{i}{\hbar}\hat{H}(t')\,dt'$$

$$\Rightarrow \int_{t_0}^{t}\frac{d\hat{u}(t',t_0)}{\hat{u}(t',t_0)} = \int_{t_0}^{t}-\frac{i}{\hbar}\hat{H}(t')\,dt'$$

$$\Rightarrow \ln\hat{u}(t,t_0) - \ln\underbrace{\hat{u}(t_0,t_0)}_{1} = \int_{t_0}^{t}-\frac{i}{\hbar}\hat{H}(t')\,dt'$$

$$\Rightarrow \underbrace{\ln\frac{\hat{u}(t,t_0)}{1}}_{\ln\hat{u}(t,t_0)} = \int_{t_0}^{t}-\frac{i}{\hbar}\hat{H}(t')\,dt'$$

$$\Rightarrow \hat{u}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')\,dt'}$$

Now, two cases

#1 $[\hat{H}(t_1), \hat{H}(t_2)] = 0 \quad (t_1 \neq t_2)$

$$\Rightarrow \hat{u}(t, t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'}$$

#2 $[\hat{H}(t_1), \hat{H}(t_2)] \neq 0 \quad (t_1 \neq t_2)$

$$\hat{u} = \hat{\mathcal{T}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \int_{t_0}^{t} \hat{H}(t')dt'\right)^n$$

Time-ordering operater.

$$= \mathbb{1} + \hat{\mathcal{T}} \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n \left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \, \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n)\right)$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \hat{H}(t_1)\cdots\hat{H}(t_n)\right)$$

where $t \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq t_0$

nested integral

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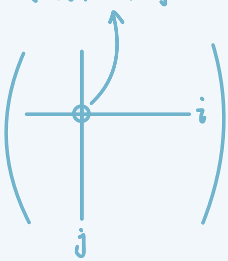
*. for the rest of obj #2. we assume $\hat{H}$ is t-independent.

1-3. Energy eigen value & eigenket.

suppose base kets $\{|a'\rangle\}$ $\quad \hat{A}|a'\rangle = a'|a'\rangle$

& $[\hat{A},\hat{H}] = 0 \Rightarrow \hat{H}|a'\rangle = \underbrace{E_{a'}}_{\text{Energy eigenvalue.}}\underbrace{|a'\rangle}_{\text{Energy eigenket.}}$

1> $\langle a_i|\hat{A}|a_j\rangle$



2> $|a_i\rangle\langle a_j| = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

3> $\sum_i \sum_j |a_i\rangle\langle a_j| \underbrace{(\langle a_i|\hat{A}|a_j\rangle)}_{\text{scalar.}}$

$= \hat{A}$

$= \sum_i \sum_j |a_i\rangle\langle a_i|\underbrace{\hat{A}|a_j\rangle}_{a_j|a_j\rangle}\langle a_j|$

$\langle a_i|\hat{A}|a_j\rangle = a_j\langle a_i|a_j\rangle = a_j\delta_{ij}$

$= \underbrace{\sum_i a_i|a_i\rangle\langle a_i|}_{\hat{A}}$

$\Rightarrow \hat{u} = e^{-\frac{i}{\hbar}\hat{H}(t-t_0)}, \quad t_0 \Rightarrow 0$

$= e^{-\frac{i}{\hbar}\hat{H}t}$, $\quad \hat{H} \to$ Energy eigenvalues?