

1-2 t-dependent Schrödinger equation

$$\begin{aligned}\hat{u}(t+dt, t_0) &= \hat{u}(t+dt, t) \hat{u}(t, t_0) \\ &= \left(1 - i \frac{\hat{H}}{\hbar} dt\right) \hat{u}(t, t_0)\end{aligned}$$

$$\Rightarrow \hat{u}(t+dt, t_0) - \hat{u}(t, t_0) = \left(-i \frac{\hat{H}}{\hbar} dt\right) \hat{u}(t, t_0)$$

$$d \hat{u}(t, t_0)$$

$$\Rightarrow \frac{\partial \hat{u}(t, t_0)}{\partial t} = \left(-i \frac{\hat{H}}{\hbar}\right) \hat{u}(t, t_0)$$

$$\Rightarrow \boxed{i\hbar \frac{\partial \hat{u}(t, t_0)}{\partial t} = \hat{H} \hat{u}(t, t_0)} \quad *$$

put $|\alpha, t_0\rangle$ on both L.H.S. & R.H.S.

$$\Rightarrow i\hbar \frac{\partial \hat{u}(t, t_0)}{\partial t} |\alpha, t_0\rangle = \hat{H} \hat{u}(t, t_0) |\alpha, t_0\rangle$$

\downarrow
 $|\alpha, t_0; t\rangle$

$$\Rightarrow \boxed{i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle = \hat{H} |\alpha, t_0; t\rangle} \quad *$$

t-dependent Schrödinger eq.

*. How to know the physical state at arbitrary time? (t)

Instead of solving $i\hbar \frac{\partial}{\partial t} |\alpha, t_0; t\rangle = \hat{H} |\alpha, t_0; t\rangle$

we solve

$$i\hbar \frac{\partial \hat{u}(t, t_0)}{\partial t} = \hat{H} \hat{u}(t, t_0)$$

Because, $\hat{u}(t, t_0) |\alpha; t_0\rangle = |\alpha, t_0; t\rangle$

$$\hat{u}(t_0 + dt, t_0) = \mathbb{1} - i \frac{\hat{H}}{\hbar} dt$$

Based on nature of \hat{H} $\left\{ \begin{array}{l} \hat{H} = \hat{H} \\ \hat{H} = \hat{H}(t') \end{array} \right.$

↳ if \hat{H} is t -independent

$$\begin{aligned} \hat{u}(t, t_0) &= \lim_{N \rightarrow \infty} \hat{u}\left(t_0 + N \frac{\Delta t}{N}, t_0\right) \\ &\quad \downarrow \\ & \quad t_0 + \Delta t \\ &= \lim_{N \rightarrow \infty} \hat{u}(t_0 + N dt, t_0) \\ &= \lim_{N \rightarrow \infty} \left[\hat{u}(t_0 + N dt, t_0 + (N-1) dt) \right. \\ &\quad \cdot \hat{u}(t_0 + (N-1) dt, t_0 + (N-2) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-2) dt, t_0 + (N-3) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-3) dt, t_0 + (N-4) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-4) dt, t_0 + (N-5) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-5) dt, t_0 + (N-6) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-6) dt, t_0 + (N-7) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-7) dt, t_0 + (N-8) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-8) dt, t_0 + (N-9) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-9) dt, t_0 + (N-10) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-10) dt, t_0 + (N-11) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-11) dt, t_0 + (N-12) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-12) dt, t_0 + (N-13) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-13) dt, t_0 + (N-14) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-14) dt, t_0 + (N-15) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-15) dt, t_0 + (N-16) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-16) dt, t_0 + (N-17) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-17) dt, t_0 + (N-18) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-18) dt, t_0 + (N-19) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-19) dt, t_0 + (N-20) dt) \\ &\quad \cdot \hat{u}(t_0 + (N-20) dt, t_0) \left. \right] \\ &= \lim_{N \rightarrow \infty} \left(\mathbb{1} - i \frac{\hat{H}}{\hbar} dt \right)^N \\ &\quad \downarrow \\ &\quad \frac{\Delta t}{N} = \frac{t - t_0}{N} \\ &= \lim_{N \rightarrow \infty} \left(\mathbb{1} - i \frac{\hat{H}}{\hbar} \cdot \frac{t - t_0}{N} \right)^N \end{aligned}$$

we set $\chi = -i \frac{\hat{H}}{\hbar} \cdot (t - t_0)$

$$\Rightarrow \lim_{N \rightarrow \infty} \left(1 + \frac{x}{N}\right)^N = e^x = e^{-i\frac{\hat{H}}{\hbar}(t-t_0)}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

similarly. $\Rightarrow \sum_{n=0}^{\infty} \frac{(-i\frac{\hat{H}}{\hbar}(t-t_0))^n}{n!}$ ← Expansion Form.

2> if \hat{H} is t -dependent. $\hat{H} = \hat{H}(t')$

$$i\hbar \frac{\partial \hat{u}(t, t_0)}{\partial t} = \hat{H} \hat{u}(t, t_0)$$

$$\downarrow$$

$$\hat{H}(t')$$

$$\Rightarrow \frac{d \hat{u}(t', t_0)}{\hat{u}(t', t_0)} = -\frac{i}{\hbar} \hat{H}(t') dt'$$

$$\Rightarrow \int_{t_0}^t \frac{d \hat{u}(t', t_0)}{\hat{u}(t', t_0)} = \int_{t_0}^t -\frac{i}{\hbar} \hat{H}(t') dt'$$

$$\Rightarrow \ln \hat{u}(t, t_0) - \ln \hat{u}(t_0, t_0) = \int_{t_0}^t -\frac{i}{\hbar} \hat{H}(t') dt'$$

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$$\Rightarrow \ln \frac{\hat{u}(t, t_0)}{\underline{1}} = \int_{t_0}^t -\frac{i}{\hbar} \hat{H}(t') dt'$$

$$\ln \hat{u}(t, t_0)$$

$$\Rightarrow \hat{u}(t, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt'}$$

Now, two cases

$$\#1 \quad [\hat{H}(t_1), \hat{H}(t_2)] = 0 \quad (t_1 \neq t_2)$$

$$\Rightarrow \hat{u}(t, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt'}$$

$$\#2 \quad [\hat{H}(t_1), \hat{H}(t_2)] \neq 0 \quad (t_1 \neq t_2)$$

$$\hat{u} = \hat{T} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt' \right)^n$$

Time-ordering operator.

$$= \mathbb{1} + \hat{T} \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \right)^n \left(\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{H}(t_1) \hat{H}(t_2) \cdots \hat{H}(t_n) \right)$$

$$= \mathbb{1} + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar} \right)^n \left(\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_{n-1}} dt_n \hat{H}(t_1) \cdots \hat{H}(t_n) \right)$$

where $t \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq t_0$

nested integral

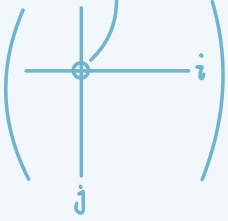
*. for the rest of obj #2. we assume \hat{H} is t -independent.

1-3. Energy eigen value & eigen ket.

suppose base kets $\{|a'\rangle\}$ $\hat{A}|a'\rangle = a'|a'\rangle$

& $[\hat{A}, \hat{H}] = 0 \Rightarrow \hat{H}|a'\rangle = E a'|a'\rangle$
 $\downarrow \quad \longmapsto$ Energy eigen ket.
 Energy eigen value.

1) $\langle a_i | \hat{A} | a_j \rangle$



$\Rightarrow |a_i\rangle\langle a_j| = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & 1 & \vdots \\ 0 & \dots & 0 \end{pmatrix}$

$\Rightarrow \sum_i \sum_j |a_i\rangle\langle a_j| \langle a_i | \hat{A} | a_j \rangle$
 $= \hat{A}$ scalar.

$= \sum_i \sum_j |a_i\rangle\langle a_i | \hat{A} | a_j \rangle \langle a_j|$
 $\frac{a_j |a_j\rangle}{a_j \langle a_i | a_j \rangle}$
 $\frac{\delta_{ij}}{a_j \langle a_i | a_j \rangle}$
 $= \sum_i a_i |a_i\rangle\langle a_i|$
 \hat{A}

$\Rightarrow \hat{u} = e^{-\frac{i}{\hbar} \hat{H} (t - \underbrace{t_0}_0)} = e^{-\frac{i}{\hbar} \hat{H} t}$
 \longmapsto Energy eigenvalues?